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Fast detection of nonlinearity and nonstationarity in short and noisy time series

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Abstract. - We introduce a statistical method to detect nonlinearity and nonstationarity in time series, that works even for short sequences and in presence of noise. The method has a discrimination power similar to that of the most advanced estimators on the market, yet it depends only on one parameter, is easier to implement and faster. Applications to real data sets reject the null hypothesis of an underlying stationary linear stochastic process with a higher confidence interval than the best known nonlinear discriminators up to date.

Natural phenomena are often studied through measures of physical observables that change over time. Hence, time series analysis is of extraordinary importance for the comprehension and the characterization of a physical process. The analysis of a time series should generally be able to detect, within some confidence level, the stochastic or deterministic nature of the underlying process, and eventually to quantify the degrees of freedom, the presence of nonlinearity or nonstationarity, and finally the predicability of the future states. Experimental time series are affected by measurement error, and they are often corrupted by unknown noise sources. In addition to this, while some processes, such as laser emissions [1] or network traffic [2], can produce a large amount of data in few hours, other natural processes, such as sunspots [3–5] or seismic events [6], may require long times of observation to obtain relevant informations. As a consequence, methods of time series analysis should be able to work on both short and long noisy series.

The surrogate data method provides a rigorous statistical approach to the nonlinear features detection of a time series [7, 8]. The method consists in formulating a null hypothesis, e.g. "the time series is generated by a linear and stationary stochastic process", an alternative hypothesis, e.g. "the time series is *not* generated by a stationary linear stochastic process", and in preparing a set of N constrained surrogate time series, generally with the same linear statistical features as the original one. One or more observables, so-called *estimators*, *discriminators*, or *dis-*

criminating statistics, are obtained from both the original time series and for the surrogates representing the null hypothesis: by performing a nonparametric test, as the rank order, observables are statistically analysed and the rejection of the null hypothesis is claimed within a certain confidence level. As pointed out in Refs. [7, 8], a large number of different measures have been considered over the years to detect nonlinearity in time series: higher order statistics, time reversal asymmetry, correlation dimension [9], largest Lyapunov exponent, nonlinear prediction error (NPE) [9–11] and Volterra-Wiener-Korenberg polynomials [12], just to mention those most commonly adopted. However, some of the measures either have low discrimination power [9], or are not able to distinguish chaos from coloured noise, as the correlation dimension [13–15], or require long time series, generally not available in the real world. In particular, the NPE, a method based on phase-space reconstruction and that requires three parameters, has the highest discrimination power on short and noisy time series, and gives either better or comparable performance than other methods [9].

In this Letter, we introduce a fast method to reject the null hypothesis of an underlying stationary linear stochastic process. The method does not require any embedding procedure for phase space reconstruction, and is based on the evaluation of the differences between the original time series and the stationary linear model that best approximates it. The kurtosis of the distribution of differences is finally used as discrimination statistics to reject the null

hypothesis. The method works well with both deterministic and stochastic series, either time continuous or discrete. It is simpler to compute and faster than other excellent discriminators, and it appears to be robust even for very short time series, highly corrupted with measurement noise. Indeed, as an application to a still open question, we will show how the method improves up to 98% the confidence level for the rejection of null hypotheses in the cases of the monthly smoothed sunspots index.

Given a l-samples univariate time series $\{s_n\}$, with $n = 1, 2, \ldots, l$, we produce a set $\{\hat{s}_n^{(i)}\}$, $i = 1, 2, ..., N$, of N surrogates of $\{s_n\}$. Each of the surrogate is a time series having the same mean, variance, density function, power spectrum and, thus, autocorrelation function as the original time series $\{s_n\}$. Different procedures to produce surrogates fulfilling several requirements have been introduced over the years [16, 17]. In particular, here, we adopt an iterative amplitude adjusting Fourier transform (IAAFT) scheme [16, 18]. Since the density function and the power spectrum of $\{s_n\}$ and $\{\hat{s}_n^{(i)}\}$ are the same, nonlinearity or nonstationarity, if present, should emerge from the analysis of the deviations from the best stationary linear approximation of the time series. The simplest stationary linear model is the autoregressive AR(p) of order p, defined as

$$x_n = a_0 + \sum_{i=1}^{p} a_i x_{n-i} + \epsilon_n \qquad (1)$$

where the integer p is the memory range of the model, and ϵ_n is a stochastic stationary process with $\langle \epsilon \rangle = 0$. To ensure the stationarity of the process, the coefficients a_i in Eq. (1) have to be chosen such that the roots of the characteristic polynomials: $1 - \sum_{i=1}^{p} a_i z^i = 0$ lie outside the unit circle. AR(p) are well known models used in time series fitting problems, and a wide literature covers the exact and efficient estimation of the coefficients a_i (see Ref. [19] for an extensive and up-to-date review). Here, we adopt a least squares approach, solved by a QR decomposition algorithm. As a result of fitting the original time series $\{s_n\}$ with an AR(p), we get the coefficients $\tilde{a}_i$, the model $\{\tilde{x}_n\}$, and the residuals series $\{\tilde{\epsilon}_n\}$, defined as $\tilde{\epsilon}_k = s_k - \tilde{x}_k$ for $k = 1, 2, ..., n$. As for the order p of the autoregressive model that best approximates the original time series $\{s_n\}$, we have chosen the one that minimizes an information criterion, according to Akaike [20] and Schwarz [21]. The estimator of linearity and stationarity we propose is the kurtosis of the residuals:

$$\mathcal{K} = \frac{\left\langle (\tilde{\epsilon} - \langle \tilde{\epsilon} \rangle)^4 \right\rangle}{\left\langle (\tilde{\epsilon} - \langle \tilde{\epsilon} \rangle)^2 \right\rangle^2} \qquad (2)$$

The same fitting procedure is repeated for each of the N surrogate series $\{\hat{s}_n^{(i)}\}$, and the obtained estimators are respectively indicated as $\hat{\mathcal{K}}^{(i)}$. Finally, the test against

the null hypothesis is based on comparing $\mathcal{K}$ to the distribution of $\hat{\mathcal{K}}^{(i)}$. We name the discriminator in Eq. 2 *autoregressive-fit residuals kurtosis* (ARK).

Summing up, the method consists, in practice, in the following steps:

1. Given a time series $\{s_n\}$, obtain the best stationary linear model that minimizes an information criterion.

2. Build the time series $\tilde{\epsilon}_n$ of residuals and evaluate the kurtosis of their density.

3. Fix a significance level α, and create $N = 2/\alpha - 1$ surrogates of $\{s_n\}$. Repeat points 1 and 2 for each surrogate time series $\{\hat{s}_n^{(i)}\}$.

4. Finally, compare the values of the discriminator in Eq. (2) obtained for $\{s_n\}$ and $\{\hat{s}_n^{(i)}\}$ with a two-sided rank order test of size α.

We have tested ARK with a wide variety of short time series, either from models and from real data sets.

Synthetic time series. We first considered stationary linear autoregressive moving-average models (ARMA), i.e. discrete null models. We performed an extensive and systematic analysis of ARK statistics, by varying both autoregressive and moving average orders. As expected, for a discriminating statistics evaluated on null models, we obtained a flat density of p-values. The same result was obtained when considering stationary linear Langevin processes, i.e. continuous null models. It follows that ARK is not biased against the null hypothesis.

We then addressed autoregressive moving-average models in nonstationary regime, and several well known nonlinear models, both in chaotic and non chaotic regime. In particular, we tested the stability of the method in discriminating nonlinearity and nonstationarity under the contamination of stochastic noise, correlated or uncorrelated, of increasing strength, by calculating the probability β to accept the null hypothesis when the alternative is true (also known as type II error). For each simulated time series with variance σ_s^2, we added artificial noise with variance σ_ϵ^2, and we studied the so-called *power* $1 - \beta$ of ARK by varying the ratio σ_ϵ/σ_s. In practice, the results obtained where the method has a power of less than 70% are considered questionable [9]. In Fig. 1 we report the power $1 - \beta$ versus the relative noise level σ_ϵ/σ_s, for different dynamical systems. The results were obtained with 10^4 rank order tests of size $\alpha = 1\%$ ($N = 199$), corresponding to a confidence level of 99%. We considered both deterministic and stochastic series, either time continuous or discrete. In the case of noise-free nonlinear or nonstationary time series, we get a discrimination power close to 100%. Even in the presence of noise, the ARK exhibits an excellent discrimination power. For all the considered dynamical systems, a sequence of $l = 1024$ samples is enough to achieve a discrimination power larger than 70%, even for relative
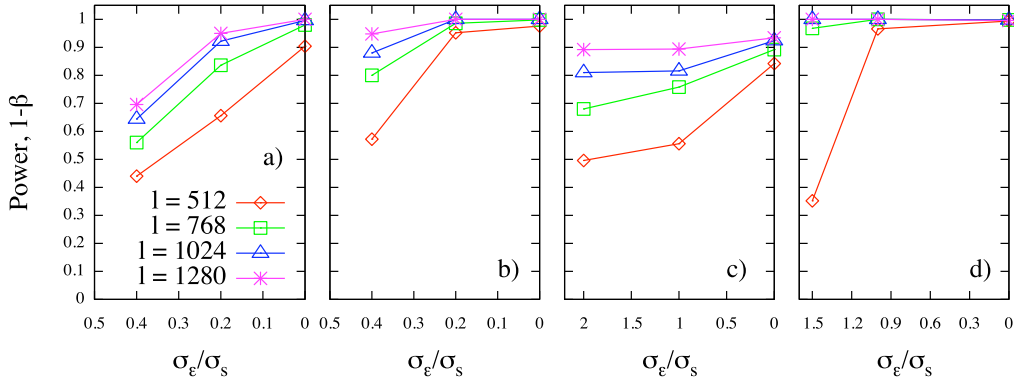
Figure 1: ARK power versus noise strength for: a) Mackey-Glass map in the chaotic regime with the addition of correlated noise,b) Ikeda map in the chaotic regime, with uncorrelated noise, c) nonstationary autoregressive moving average (ARMA) with correlated noise, d) Lorenz model in chaotic regime with uncorrelated noise.

noise levels as large as 40% in the case of maps, and as large as 150% in the case of the nonstationary ARMA and of the Lorenz system. Notice that adding a relative noise level of 150% to a time series corresponds to super-imposing a noise amplitude that is 1.5 times the intrinsic fluctuations of the original series. Indeed, in the case of a chaotic dynamics, such as the Lorenz system reported in panel d), it is possible to reach a discrimination power close to 100% also for shorter length ($l = 768$). This result is particularly relevant if compared against the result from NPE: in presence, for instance, of a relative noise level of 150%, the phase space of the Lorenz model is not well re-constructed through the delay embedding method, and, in fact, a power of no more than 30% is achieved. Hence, the results in Fig. 1 show that our procedure is suited for the analysis of real datasets, generally noisy ($\sigma_\epsilon/\sigma_s < 1$) time series of few thousands of samples.

Real data sets. The first nonlinear experimental time series we have considered is a sample of $l = 1000$ values of the intensity of a Far Infrared Laser (FIR) in a chaotic state (see Ref. [1]). In Fig. 2a we show the rank ordered ARK values for both the FIR time series and 1000 of its surrogates: as expected, the rank order test rejects the null hypothesis with 99.8% confidence level.

The second nonlinear experimental time series we have considered is a sample of $l = 4096$ values of intracranial EEG recordings during epileptic seizures (see Ref. [22]). In Fig. 2b we show the rank ordered ARK values for both the EEG time series and 1000 of its surrogates: as expected, the rank order test rejects the null hypothesis with 99.8% confidence level.

As third application to real databases, the discrimination power of ARK is tested against time series of sunspots index. In particular, we consider the monthly smoothed sunspots index from 1749 to 2008, and the 11-years aver-aged sunspots index reconstructed up to the past 11,400
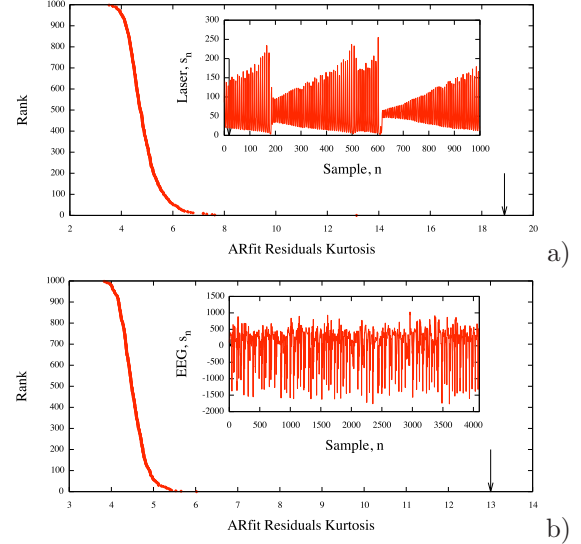

Figure 2: Rank order test (1000 surrogates) for the Far Infrared Laser intensity time series (a) and for an EEG recording during epileptic seizure (b). The arrow indicates the value of ARK for the original time series, dots indicate the rank ordered values of ARK for their surrogates.

years. The sunspots index, introduced by Wolf in 1848, is strongly related to the solar cycle discovered by Schwabe in 1843. Such series shows strong irregularities, partially explained by magnetohydrodynamics of dynamo, and a quasi-periodic behavior. However, the real nature of solar cycle is still debated, as shown in the following. Gur-buz and Beck [23] claimed that the dynamics of successive sunspot maxima is low-dimensional with features similar to the intermittent logistic map. Barnes [24] proposed an ARMA(2,2) model mapped by a nonlinear function to reproduce the solar cycle with its statistical features and its irregular and unpredictable behavior. In conflict

with this result, a randomly driven nonlinear oscillator was proposed for the first time by Paluŝ and Novotná [3] by using the mutual dependence of the instantaneous amplitude and frequency of sunspot series as discriminator in a surrogate data test, to reject the null hypothesis of an underlying Barnes model [24]. The mapping to a scalar time series from the spatio-temporal magnetic field described by a nonlinear, eventually stochastically, driven partial differential equation for the magnetohydrodynamic dynamo, was not excluded from the results [4,5]. Sunspot irregularities were attributed to the stochastic fluctuation in one of the parameters of a Van Der Pol nonlinear oscillator describing the irregular periodic magnetic field [25], while recently, an overembedding approach introduced for modeling and prediction of nonstationary systems was successfully applied to the series with high precision [26].
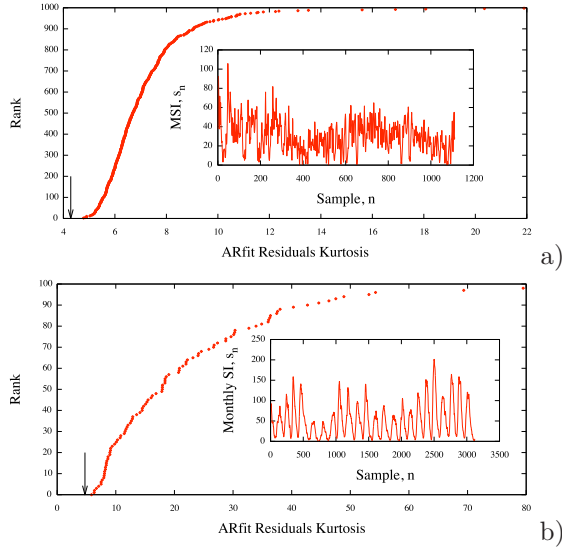

Figure 3: Rank order test for the sunspots index, averaged on 11-years, reconstructed up to the past 11,400 years against 1000 Barnes null models (a), and for the monthly smoothed sunpost index from 1749 to 2008 and 100 surrogates (b). The arrow indicates the value of ARK for the original time series, dots indicate the rank ordered values of ARK for their surrogates.

We considered the series of $l = 3123$ samples for the monthly smoothed sunspots index, released by the Solar Influences Data analysis Center [27] from 1749 to 2008, and the historical series of $l = 1113$ samples for the sunspots index, averaged over 11-years, reconstructed up to the past 11,400 years (MSI) through indirect methods [28,29]. Standard estimators as the Lyapunov exponent, the correlation dimension, and the increase of the prediction error with the prediction horizon can lead to spurious results when applied to short time series, as shown for instance in Ref. [25] and references therein.

Here, we have applied ARK both to the $l = 1113$ and to the $l = 3123$ series, performing surrogate tests against different null models. If a null hypothesis can not be rejected, the sunspots index time series is not distinguish-

able from the null model: thus either it is well described by the model or the used statistics has not enough discrimination power. We started by considering the short time series and Barnes time series as null models. Fig. 3a shows the rank ordered ARK values for the historical series and 1000 of its nulls: we reject the null hypothesis of an underlying Barnes process with 99.8% confidence level. Successively, the long time series was tested against IAAFT surrogates. Fig. 3b shows the rank ordered ARK values for the monthly series and 100 of its surrogates: we reject the null hypothesis of an underlying stationary linear stochastic process with 98% confidence level.

We verified that one of the best test statistics up to date [9], namely the nonlinear prediction error, gives similar results. Indeed, we emphasize that NPE makes use of three parameters, the lag time ν, the embedding dimension M and the size of neighbourhoods, while ARK does not need any phase space reconstruction, and only uses one parameter. In addition to this, our numerical experiments on simulated time series show that, excluding the time for the estimation of needed parameters, ARK complexity is generally $\mathcal{O}(l)$, while NPE complexity is $\mathcal{O}(l^2)$ [30], where l is the time series length. Fig. 4 shows the CPU time in seconds versus the length of the series, for both NPE and ARK applied to Lorenz models of increasing number of samples: complexity is $\mathcal{O}(l^{1.96\pm0.02})$ for the former and $\mathcal{O}(l^{0.96\pm0.03})$ for the latter.
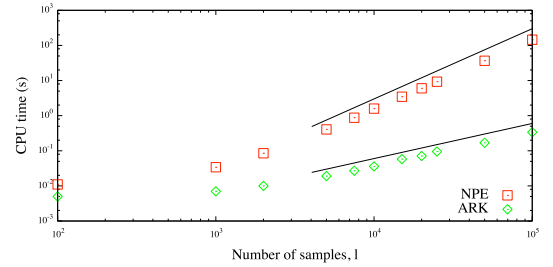

Figure 4: CPU time in seconds versus the length of the series, for both NPE and ARK applied to Lorenz models of increasing number of samples.

In this Letter, we have introduced ARK analysis, a new procedure to detect nonlinearity and nonstationarity in a time series. We have tested the discrimination power of ARK on a wide variety of short and noisy time series, either from models and from real data sets. A series of the far infrared laser emission in a chaotic state and a sample of the intracranial EEG recordings during epileptic seizure, have been analyzed: in both case our procedure was able to detect, with high confidence level, the presence of nonlinearity. Using Barnes models, supposed to replicate the behavior of the annual sunspots index, we have obtained a statistically significant rejection of the null hypothesis of an underlying stationary linear stochastic process, possibly mapped by a nonlinear function, for the historical time series. Using surrogates, we have obtained a

statistically significant rejection of the null hypothesis of an underlying stationary linear stochastic process, for the monthly smoothed sunspot index time series. Although no particular model for sunspots index has been proposed here, the presented results are an important step for the comprehension of the underlying solar cycle mechanisms. We believe that our method can be successfully applied to other interesting real-world time series whose underlying dynamics is still debated.

References

[1] Hubner U., Abraham N.B. and Weiss C.O., *Phys. Rev. A* , **40** (1989) 6354.

[2] Meloni S., Gomez-Gardenes J., Latora V. and Moreno Y., *Phys. Rev. Lett* , **100** (2008) 208701.

[3] Paluŝ M. and Novotná D., *Phys. Rev. Lett.* , **83** (1999) 3406.

[4] Timmer J., *Phys. Rev. Lett.* , **85** (2000) 2647.

[5] Paluŝ M., *Phys. Rev. Lett.* , **85** (2000) 2648.

[6] Mega M.S., Allegrini P., Grigolini P., Latora V., Palatella L., Rapisarda A. and Vinciguerra S., *Phys. Rev. Lett* , **90** (2003) 188501.

[7] Theiler J., Eubank S., Longtin A., Galdrikian B. and Farmer J.D., *Physica D* , **58** (1992) 77.

[8] Prichard D. and Theiler J., *Phys. Rev. Lett.* , **73** (1994) 951.

[9] Schreiber T. and Schmitz A., *Phys. Rev. E* , **55** (1997) 5443.

[10] Farmer J.D. and Sidorowich J.J., *Phys. Rev. Lett.* , **59** (1987) 845.

[11] Sugihara G. and May R., *Nature* , **344** (1990) 734.

[12] Barahona M. and Poon C., *Nature* , **381** (1996) 215.

[13] Osborne A.R. and Provenzale A., *Physica D* , **35** (1989) .

[14] Provenzale A., Osborne A.R. and Soj R., *Physica D* , **35** (1991) .

[15] Provenzale A., Smith L.A., Vio R. and Murante G., *Physica D* , **58** (1992) 31.

[16] Schreiber T., *Phys. Rev. Lett.* , **80** (1998) 2105.

[17] Schreiber T. and Schmitz A., *Physica D* , **142** (2000) 346.

[18] Schreiber T. and Schmitz A., *Phys. Rev. Lett.* , **77** (1996) 635.

[19] Brockwell P. and Davis R., *Time series: theory and methods* (Springer Verlag) 2009.

[20] Akaike H., *IEEE Trans. on Aut. Control A*, **C−19** (1974) .

[21] Schwarz G., *The annals of statistics* , **6** (1978) 461.

[22] Andrzejak R., Lehnertz K., Mormann F., Rieke C., David P. and Elger C., *Phys. Rev. E* , **64** (2001) 61907.

[23] Gurbuz G. and Beck C., *Int. J. Bifurcation Chaos Appl. Sci. Eng.* , **5** (1995) 1213.

[24] Barnes J., Sargent H. and Tryon P., *Proc. Conf. Ancient Sun* , (1980) 159.

[25] Mininni P.D., Gómez D.O. and Mindlin G.B., *Phys. Rev. Lett.* , **85** (2000) 5476.

[26] Verdes P.F., Granitto P.M. and Ceccatto H.A., *Phys. Rev. Lett.* , **96** (2006) 118701.

[27] SIDC-team, *The International Sunspot Number* (1749–2008).

[28] Solanki S.K., Usoskin I.G., Kromer B., Schüssler M. and Beer J., *Nature* , **431** (2004) 1084.

[29] Usoskin I.G., Solanki S.K., Schüssler M., Mursula K. and Alanko K., *Phys. Rev. Lett.* , **91** (2003) 211101.

[30] Hegger R., Kantz H. and Schreiber T., *Chaos* , **9** (1999) 413.